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On some Tutte polynomial sequences in the square lattice

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ABSTRACT

Let $T(L_{m,n}; x, y)$ be the Tutte polynomial of the square lattice $L_{m,n}$, for integers $m, n \in \mathbb{Z}_{>0}$. Using a family of Tutte polynomial inequalities established by the author in a previous work, we study the analytical properties of the sequences $(T(L_{m,n}; x, y)^{1/mn}\colon n \in \mathbb{Z}_{>0})$ for a fixed $m \in \mathbb{Z}_{>0}$, and $(T(L_{n,n}; x, y)^{1/n^2}\colon n \in \mathbb{Z}_{>0})$, in the region $x, y \geqslant 1$. We show that these sequences are monotonically increasing when $(x-1)(y-1) > 1$. We also compute lower bounds for these limits when $(x-1)(y-1) > 1$, and upper bounds when $(x-1)(y-1) < 1$. At the point $(x = 2,\ y = 1)$, where the Tutte polynomial is known to count the number of forests, we compute $\lim_{n\to\infty} T(L_{n,n}; 2, 1)^{1/n^2} \leqslant 3.705603$, which improves upon the previous best upper bound of 3.74101 obtained by Calkin, Merino, Noble and Noy (2003).

1. Introduction

The evaluation of the Tutte polynomial of the square lattice at various points on the two-dimensional real plane is of interest in combinatorics and statistical physics. For example, in the region $x, y \geqslant 1$, the Tutte polynomial counts the number of spanning trees, forests and connected subgraphs of the lattice at the points $(1, 1)$, $(2, 1)$ and $(1, 2)$, respectively, while it is computationally equivalent to the ferromagnetic versions of the Ising model of statistical physics at points along the positive branch of the curve $(x-1)(y-1) = 2$, and Potts model along the positive branch of the curve $(x-1)(y-1) = q$ for integers $q > 0$ [21, Chapters 3 and 4].

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Let $G(V, E)$ be a graph. For $X \subseteq E$, we use $\langle X\rangle$ to denote the subgraph of G with edge set X, and vertex set, $V(\langle X\rangle) \subseteq V$, containing only those vertices of V that are incident with at least one edge in X. Note that our definition of $\langle X\rangle$ precludes isolated vertices from its vertex set. The *rank function* of G, $\rho : 2^E \to \mathbb{Z}_{\geqslant 0}$, is then defined for all $X \subseteq E$ by

$$\rho(X) = \bigl|V\bigl(\langle X\rangle\bigr)\bigr| - k\bigl(\langle X\rangle\bigr), \tag{1}$$

where $k(\langle X\rangle)$ is the number of connected components of $\langle X\rangle$. The *Tutte polynomial* of G is

$$T(G; x, y) = \sum_{X \subseteq E} (x-1)^{\rho(E)-\rho(X)}(y-1)^{|X|-\rho(X)}. \tag{2}$$

For positive integers m, n, the *square lattice* is the graph $L_{m,n}$, with vertex set $V = \{(i, j)\colon 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}$, where vertices (i, j) and (k, l) are connected by an edge if and only if $|i-k| + |j-l| = 1$. Note that $L_{m,n}$ has mn vertices and $2mn - (m+n)$ edges. Also, the graphs $L_{m,n}$ and $L_{n,m}$ are isomorphic, and hence $T(L_{m,n}; x, y) = T(L_{n,m}; x, y)$.

For positive integers m, n and $(x, y) \in \mathbb{R}^2$, let

$$a_{m,n}(x, y) = T(L_{m,n}; x, y)^{1/mn} \quad \text{and} \tag{3a}$$

$$b_n(x, y) = T(L_{n,n}; x, y)^{1/n^2}. \tag{3b}$$

Wu [22], Shrock and Wu [19] and Chang and Shrock [7] calculated the limit of the sequence $(b_n(1, 1)\colon n \in \mathbb{Z}_{>0})$ to be $3.209912\ldots$, where the Tutte polynomial is known to count the number of spanning trees. Biggs ([2,3]) studied the sequence $(b_n(x, y)\colon n \in \mathbb{Z}_{>0})$ at the points $(1-\lambda, 0)$, where the Tutte polynomial is known to count the number of λ-colorings. Shrock [18] and Chang and Shrock [5] computed exact values of the limits of the sequences $(a_{2,n}(x, y)\colon n \in \mathbb{Z}_{>0})$ and $(a_{3,n}(x, y)\colon n \in \mathbb{Z}_{>0})$ for all $(x, y) \in \mathbb{R}^2$. Merino and Welsh [15] and Calkin, Merino, Noble and Noy [4] gave upper and lower bounds for the limit of the sequence $(b_n(x, y)\colon n \in \mathbb{Z}_{>0})$ at the points $(2, 1)$ and $(2, 0)$, where the polynomial counts the number of forests and acyclic orientations of a graph, respectively. Chang and Shrock [6] calculated the limits of the sequence $(a_{m,n}(2, 0)\colon n \in \mathbb{Z}_{>0})$ for integers m in the interval $1 \leqslant m \leqslant 12$.

We are concerned with the limits of the sequences $(a_{m,n}(x, y)\colon n \in \mathbb{Z}_{>0})$ for a fixed positive integer m, and $(b_n(x, y)\colon n \in \mathbb{Z}_{>0})$ when $x, y \geqslant 1$. The existence of the limits in this region has been established previously by Grimmett [10]. Traditional approaches to obtaining bounds for such limits have used transfer-matrices (see [3,4]). In this paper, we use a family of inequalities, introduced in [14], to obtain one-sided bounds for these limits in the region $x, y \geqslant 1$.

It is also worth noting that while the first few terms of $(a_{m,n}(x, y)\colon n \in \mathbb{Z}_{>0})$ for small fixed positive integers m, and $(b_n(x, y)\colon n \in \mathbb{Z}_{>0})$ suggest as much, so far there is no known proof that these sequences are monotonically increasing when $x, y \geqslant 1$. In one of our results in this paper, we establish this when $x, y > 1$ and $(x-1)(y-1) > 1$.

For positive integers m and all $x, y \geqslant 1$, let

$$\alpha_m(x, y) = \lim_{n\to\infty} a_{m,n}(x, y) \quad \text{and} \tag{4a}$$

$$\beta(x, y) = \lim_{n\to\infty} b_n(x, y). \tag{4b}$$

Also, for positive integers k, m, n such that $k < n$, and $(x, y) \in \mathbb{R}^2$, let

$$c^k_{m,n}(x, y) = \left(\frac{T(L_{m,n}; x, y)}{T(L_{m,k}; x, y)}\right)^{1/m(n-k)} \quad \text{and} \tag{5a}$$

$$\gamma^k_n(x, y) = \left(\frac{\alpha_n(x, y)^n}{\alpha_k(x, y)^k}\right)^{1/(n-k)}. \tag{5b}$$

Further, let $H_{>1} = \{(x, y)\colon (x, y) \in \mathbb{R}^2_{>1},\ (x-1)(y-1) > 1\}$ and $H_{<1} = \{(x, y)\colon (x, y) \in \mathbb{R}^2_{\geqslant 1},\ (x-1)(y-1) < 1\}$. Our main results can be summarized as follows.

- For points (x, y) in the region $H_{>1}$:
  1. For fixed $k, m \in \mathbb{Z}_{>0}$, the two sequences $(a_{m,n}(x, y)\colon\ n \in \mathbb{Z}_{>0})$ and $(c^k_{m,n}(x, y)\colon\ n \in \mathbb{Z}_{>k})$ are monotonically increasing, and converge to the limit $\alpha_m(x, y)$.
  2. For a fixed $k \in \mathbb{Z}_{>0}$, the three sequences $(b_n(x, y)\colon\ n \in \mathbb{Z}_{>0})$, $(\alpha_m(x, y)\colon\ m \in \mathbb{Z}_{>0})$ and $(\gamma^k_n(x, y)\colon\ n \in \mathbb{Z}_{>k})$ are monotonically increasing, and converge to the limit $\beta(x, y)$.
- For points (x, y) in the region $H_{<1}$:
  1. For a fixed $m \in \mathbb{Z}_{>0}$, the sequence $(c^1_{m,n}(x, y)\colon\ n \in \mathbb{Z}_{>1})$ is monotonically decreasing, and converges to the limit $\alpha_m(x, y)$.
  2. The sequence $(\gamma^1_n(x, y)\colon\ n \in \mathbb{Z}_{>1})$ is monotonically decreasing, and

$$\lim_{n\to\infty} \gamma^1_n(x, y) \geqslant \beta(x, y).$$

Since every term of a monotonic sequence is a bound on its limit, we can use the above results to compute upper (lower) bound for the limits $\alpha_m(x, y)$ and $\beta(x, y)$ in the region $H_{<1}$ (respectively, $H_{>1}$) from known Tutte polynomial evaluations of small finite lattices. Specifically, at the point $(x = 2, y = 1) \in H_{<1}$, our results show $\beta(2, 1) \leqslant 3.705603$, improving upon the previous best upper bound of 3.74101 obtained by Calkin, Merino, Noble and Noy [4]. (Note that [4] uses the notation $f(n)$ for our $b_n(2, 1)$.)

The rest of the paper is organized as follows. The next section introduces some Tutte polynomial inequalities for square lattices in the region $x, y \geqslant 1$ that form the basis of our results. Section 3 is a collection of some simple and useful convergence results valid for all $x, y \geqslant 1$. Section 4 discusses the behavior of the sequences $(a_{m,n}(x, y)\colon\ n \in \mathbb{Z}_{>0})$ for fixed $m \in \mathbb{Z}_{>0}$, and $(b_n(x, y)\colon\ n \in \mathbb{Z}_{>0})$ in the region $H_{>1}$, and computing lower bounds for their limits in this region, while Section 5 discusses these sequences and upper bounds for their limits in the region $H_{<1}$. Section 7 concludes with a discussion on open questions. We use the notation $G \cong H$ to denote graph G is isomorphic to H.

2. Some properties of $T(L_{m,n}; x, y)$ when $x, y \geqslant 1$

Our results depend on a series of Tutte polynomial inequalities for lattice graphs.

For integers p, q such that $1 \leqslant p \leqslant q \leqslant n$, we use $L_{m,n}\langle p : q\rangle$ to denote the subgraph of the lattice $L_{m,n}$ induced by the vertex set $\{(i, j)\colon\ 1 \leqslant i \leqslant m,\ p \leqslant j \leqslant q\}$. It can be easily checked that $L_{m,n}\langle p : q\rangle \cong L_{m,q-p+1}$.

We begin with an inequality result for the entire region $x, y \geqslant 1$. Note from (2) that for all graphs $G(V, E)$, $F \subseteq E$ and $x, y \geqslant 1$, we have $T(G; x, y) \geqslant T(\langle F\rangle; x, y)$.

Lemma 1. *Let $m, n_1, n_2, n_3 \in \mathbb{Z}_{>0}$ such that and $n_1 = n_2 + n_3$. Then for all $x, y \geqslant 1$,*

$$T(L_{m,n_1}; x, y) \geqslant T(L_{m,n_2}; x, y) \cdot T(L_{m,n_3}; x, y).$$

Proof. Let E be the edge set of L_{m,n_1}. Let $X \subseteq E$ be the set of m edges connecting vertices (i, n_2) and $(i, n_2 + 1)$ for all $1 \leqslant i \leqslant m$. Clearly the subgraph $\langle E \setminus X\rangle$ contains the two components, L_{m,n_2} and $L_{m,n_1}\langle n_2 + 1 : n_1\rangle \cong L_{m,n_3}$. Hence,

$$T(L_{m,n_1}; x, y) \geqslant T\big(\langle E \setminus X\rangle; x, y\big) = T(L_{m,n_2}; x, y) \cdot T(L_{m,n_3}; x, y). \qquad \square$$

The next inequality is a consequence of a well-known correlation property of Tutte polynomials for all matroids in the region $H_{>1}$, first shown by Seymour and Welsh [17], and restated in the following equivalent form by Mani [14].

Theorem 2. *(See Seymour and Welsh [17, Theorem 5.14].) Let G be a graph with edge set E and rank function ρ. Also let $E_1, E_2 \subseteq E$, and $k = \rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2)$. Then for all $(x, y) \in H_{>1}$,*

$$(x-1)^k \cdot T\big(\langle E_1 \cup E_2\rangle; x, y\big) \cdot T\big(\langle E_1 \cap E_2\rangle; x, y\big) \geqslant T\big(\langle E_1\rangle; x, y\big) \cdot T\big(\langle E_2\rangle; x, y\big).$$

One consequence of Theorem 2 for square lattices will be of particular interest to us in this paper.

Corollary 3. *Let m, n_1, n_2, n_3, n_4 be positive integers such that $n_1 \geqslant \max\{n_2, n_3\}$ and $n_1 + n_4 = n_2 + n_3$. Then for all $x, y \geqslant 1$ such that $(x-1)(y-1) > 1$,*

$$T(L_{m,n_1}; x, y) \cdot T(L_{m,n_4}; x, y) \geqslant T(L_{m,n_2}; x, y) \cdot T(L_{m,n_3}; x, y).$$

Proof. Let E be the edge set of L_{m,n_1}. Let E_1 be the edges of the subgraph $L_{m,n_1}\langle 1 : n_2\rangle$, and E_2 the edges of the subgraph $L_{m,n_1}\langle n_1 - n_3 + 1 : n_1\rangle$. It is easy to see that $\langle E_1 \cap E_2\rangle = L_{m,n_1}\langle n_1 - n_3 + 1 : n_2\rangle \cong L_{m,n_4}$, and $\rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2) = 0$. Since $\langle E_1\rangle = L_{m,n_2}$ and $\langle E_2\rangle \cong L_{m,n_3}$, the result now follows from Theorem 2. □

It is an open question if there is an inequality analogous to Theorem 2 in the region $H_{<1}$ for graphic matroids. Nevertheless, we can obtain a weaker inequality for this region. We first need the following definitions.

Recall that the rank function of any graph with edge set E is known to satisfy the *submodular property*, which states that for all $X, Y \subseteq E$, $\rho(X \cup Y) + \rho(X \cap Y) \leqslant \rho(X) + \rho(Y)$ [16, p. 23]. We extend this property by defining R-submodularity for any $R \subseteq E$.

Definition 4 *(Graph R-submodularity).* (See [14].) Let G be a graph with edge set E and rank function ρ. For $R \subseteq E$, we say disjoint sets $P_1, P_2 \subseteq E \setminus R$ are *R-submodular in G* if there exists a bijection $\pi : 2^R \to 2^R$ such that for all $C \subseteq R$, $\rho(P_1 \cup P_2 \cup C) + \rho(R \setminus C) \leqslant \rho(P_1 \cup \pi C) + \rho(P_2 \cup R \setminus \pi C)$. We call π an *R-submodular bijection* of the ordered pair (P_1, P_2) in G.

We note that graph R-submodularity is equivalent to a special case of the rank domination property in matroids defined in [14].

As an example, in any graph with edge set E, all disjoint pairs $E_1, E_2 \subseteq E$ are $\emptyset$-submodular, which is equivalent to the submodular property of its rank function. It can also be readily checked that in any graph $G(V, E)$, all $E_1 \subseteq E$ and $\emptyset$ are R-submodular in G for all $R \subseteq E \setminus E_1$. However, there are also known examples of graphs $G(V, E)$ and mutually disjoint $E_1, E_2, R \subseteq E$ such that E_1 and E_2 are not R-submodular in G [14, Example 3.3].

Definition 5 *(R-family of graph minors).* (See [14].) Let G be a graph with edge set E. Given an $R \subseteq E$, we define the *R-family of minors* of G, $\mathcal{MF}(G, R)$, to be

$$\mathcal{MF}(G, R) = \big\{G/C \setminus (R \setminus C):\ C \subseteq R\big\}. \tag{6}$$

That is, the R-family of minors is the set of all minors of G obtained by deleting or contracting every element in R.

The following theorem was proved for the region $H_{<1}$ in [14].

Theorem 6. *(See Mani [14].) Let G be a graph with edge set E and rank function ρ. Also let $E_1, E_2 \subseteq E$, and $k = \rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2)$. If for all $P_1 \subseteq E_1 \setminus E_2$, $P_2 \subseteq E_2 \setminus E_1$, $R \subseteq E_1 \cap E_2$ and minors $J \in \mathcal{MF}(G, (E_1 \cap E_2) \setminus R)$, the sets P_1 and P_2 are R-submodular in J, then*

$$(x-1)^k \cdot T\big(\langle E_1 \cup E_2\rangle; x, y\big) \cdot T\big(\langle E_1 \cap E_2\rangle; x, y\big) \leqslant T\big(\langle E_1\rangle; x, y\big) \cdot T\big(\langle E_2\rangle; x, y\big),$$

for all $(x, y) \in H_{<1}$.

We give a proof of the following special case of R-submodularity for square lattices in Appendix A.

Lemma 7. *Let E be the edge set of the lattice $L_{m,n}$, and for a fixed integer k in the interval $1 \leqslant k \leqslant n$, let E_k be the edges of the subgraph $L_{m,n}\langle k : k\rangle$. Also let E_1 and E_2 be the edge sets of the subgraphs $L_{m,n}\langle 1 : k\rangle$ and*

$L_{m,n}\langle k:n\rangle$, respectively. Then for all $P_1 \subseteq E_1 \setminus E_2$, $P_2 \subseteq E_2 \setminus E_1$, $R \subseteq E_k$ and minors $J \in \mathcal{MF}(L_{m,n}, E_k \setminus R)$, the sets P_1 and P_2 are R-submodular in J.

Theorem 8. *Let m, n_1, n_2, n_3 be positive integers such that $n_1 \geqslant \max\{n_2, n_3\}$ and $n_1 + 1 = n_2 + n_3$. Then for all $x, y \geqslant 1$ such that $(x-1)(y-1) < 1$,*

$$T(L_{m,n_1}; x, y) \cdot T(L_{m,1}; x, y) \leqslant T(L_{m,n_2}; x, y) \cdot T(L_{m,n_3}; x, y).$$

Proof. Let E be the edge set of L_{m,n_1}. Also let E_1 and E_2 be the edges of the subgraphs $L_{m,n_1}\langle 1:n_2\rangle$ and $L_{m,n_1}\langle n_2:n_1\rangle$, respectively. Then, we have $\langle E_1 \cap E_2\rangle = L_{m,n_1}\langle n_2:n_2\rangle \cong L_{m,1}$ and $\rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2) = 0$. Since $\langle E_1\rangle = L_{m,n_2}$ and $\langle E_2\rangle \cong L_{m,n_3}$, the result follows from Theorem 6 and Lemma 7. $\square$

However, at the point $(1,1) \in H_{<1}$, we get a stronger inequality due to a result by Feder and Mihail [9] for the class of regular matroids, which includes all graphs. This result can be stated in the following equivalent form for graphs. (Note the similarity with Theorem 2.)

Theorem 9. *(See Feder and Mihail [9, Theorem 2.1].) Let G be a graph with edge set E and rank function ρ. Then for all $E_1, E_2 \subseteq E$ such that $\rho(E_1) + \rho(E_2) - \rho(E_1 \cup E_2) - \rho(E_1 \cap E_2) = 0$,*

$$T\big(\langle E_1 \cup E_2\rangle; 1, 1\big) \cdot T\big(\langle E_1 \cap E_2\rangle; 1, 1\big) \leqslant T\big(\langle E_1\rangle; 1, 1\big) \cdot T\big(\langle E_2\rangle; 1, 1\big).$$

The following consequence of this result can be proved using arguments similar to those used in the proof of Corollary 3.

Corollary 10. *Let m, n_1, n_2, n_3, n_4 be positive integers such that $n_1 \geqslant \max\{n_2, n_3\}$ and $n_1 + n_4 = n_2 + n_3$. Then,*

$$T(L_{m,n_1}; 1, 1) \cdot T(L_{m,n_4}; 1, 1) \leqslant T(L_{m,n_2}; 1, 1) \cdot T(L_{m,n_3}; 1, 1).$$

3. Results for all $x, y \geqslant 1$

Throughout this section, we assume $x, y \geqslant 1$.

Our first result is a simple extension of a useful observation in [4, Section 7], where it was shown that $\beta(2,0) = \beta(0,2)$ and $\beta(2,1) = \beta(1,2)$. Indeed that argument can readily be applied for all $x, y \geqslant 1$ to get the following.

Lemma 11. *For all $x, y \geqslant 1$, $\beta(x, y) = \beta(y, x)$.*

Proof. Let $L^*_{n,n}$ be the dual graph of the lattice $L_{n,n}$. It can be checked that $L^*_{n,n}$ contains $L_{n-1,n-1}$ as a subgraph. Thus, when $x, y \geqslant 1$, $T(L_{n,n}; x, y) = T(L^*_{n,n}; y, x) \geqslant T(L_{n-1,n-1}; y, x)$, and hence,

$$\beta(x, y) = \lim_{n\to\infty}\big(T(L_{n,n}; x, y)\big)^{1/n^2} \geqslant \lim_{n\to\infty}\big(T(L_{n-1,n-1}; y, x)\big)^{1/n^2} = \beta(y, x).$$

Now setting $x = y$, $y = x$ reverses the direction of the inequality, and hence the result. $\square$

We next establish some convergence results for the double sequence $(c^k_{m,n}(x, y)\colon\ m, n \in \mathbb{Z}_{>0}, n > k)$ for a fixed $k \in \mathbb{Z}_{>0}$.

Lemma 12. *For fixed $k, m \in \mathbb{Z}_{>0}$, the sequence $(c^k_{m,n}(x, y)\colon\ n \in \mathbb{Z}_{>k})$ converges to the limit $\alpha_m(x, y)$.*

Proof. From (5a),

$$\begin{aligned}\lim_{n\to\infty} c_{m,n}^k(x,y) &= \lim_{n\to\infty}\left(\frac{T(L_{m,n};x,y)}{T(L_{m,k};x,y)}\right)^{1/m(n-k)}\\ &= \lim_{n\to\infty} T(L_{m,n};x,y)^{1/m(n-k)}\\ &= \lim_{n\to\infty} T(L_{m,n};x,y)^{1/mn}\cdot \lim_{n\to\infty} T(L_{m,n};x,y)^{k/mn(n-k)}\\ &= \alpha_m(x,y). \qquad \square\end{aligned}$$

Lemma 13. *For fixed $k,n\in\mathbb{Z}_{>0}$ such that $n>k$, the sequence $(c_{m,n}^k(x,y)\colon m\in\mathbb{Z}_{>0})$ converges to the limit $\gamma_n^k(x,y)$.*

Proof. From (5a),

$$\begin{aligned}\lim_{m\to\infty} c_{m,n}^k(x,y) &= \left(\frac{\lim_{m\to\infty} T(L_{m,n};x,y)^{1/m}}{\lim_{m\to\infty} T(L_{m,k};x,y)^{1/m}}\right)^{1/(n-k)}\\ &= \left(\frac{\alpha_n(x,y)^n}{\alpha_k(x,y)^k}\right)^{1/(n-k)} = \gamma_n^k(x,y). \qquad \square\end{aligned}$$

For fixed $k,m,p\in\mathbb{Z}_{>0}$ such that $k<p\leqslant m$, let

$$\xi_m^{p,k}(x,y) = \alpha_q(x,y)^{q/m}\cdot\left(\frac{\alpha_p(x,y)^p}{\alpha_k(x,y)^k}\right)^{(m-q)/m(p-k)}, \tag{7}$$

where $q\equiv m \pmod{p-k}$ and $k\leqslant q\leqslant p-1$. We note that given $p,k\in\mathbb{Z}_{>0}$ such that $k<p\leqslant m$, it is straightforward to check that either $q=(p-k)\lfloor k/(p-k)\rfloor + (m \bmod p-k)$ or $q=(p-k)\lceil k/(p-k)\rceil + (m \bmod p-k)$, and thus can be quickly computed. We will use $\xi_m^{p,k}(x,y)$ in Sections 4.3 and 5.3 to compute bounds for $\alpha_m(x,y)$. Here, we establish a convergence result for the sequence $(\xi_m^{p,k}(x,y)\colon m\in\mathbb{Z}_{\geqslant p})$, for fixed $p,k\in\mathbb{Z}_{>0}$ such that $p>k$.

Lemma 14. *For a fixed $k,p\in\mathbb{Z}_{>0}$ such that $p>k$, the sequence $(\xi_m^{p,k}(x,y)\colon m\in\mathbb{Z}_{\geqslant p})$ converges to the limit $\gamma_p^k(x,y)$.*

Proof. The result is straightforward from (7) and (5b). $\square$

4. Results in $H_{>1}$

In this section, we study the properties of the sequences $(a_{m,n}(x,y)\colon n\in\mathbb{Z}_{>0})$ and $(b_n(x,y)\colon n\in\mathbb{Z}_{>0})$ in the region $H_{>1}$. We begin by establishing some monotonicity and convergence results for these sequences, and then obtain efficiently computable lower bounds for their limits. Throughout this section, we assume $(x,y)\in H_{>1}$.

4.1. Monotonicity and convergence results

Proposition 15. *For a fixed $m\in\mathbb{Z}_{>0}$, the sequence $(a_{m,n}(x,y)\colon n\in\mathbb{Z}_{>0})$ is monotonically increasing.*

Proof. Note that from (3a) it is enough to prove for integers $n>0$,

$$T(L_{m,n+1};x,y)^n \geqslant T(L_{m,n};x,y)^{n+1}. \tag{8}$$

We use induction on n. When $n = 1$, (8) follows from Lemma 1 using $n_1 = 2$ and $n_2 = n_3 = 1$. Let $p \in \mathbb{Z}_{>0}$ and assume (8) is valid for all $n \leqslant p$.

Now, suppose $n = p + 1$. Using $n_1 = p + 2$, $n_2 = n_3 = p + 1$ and $n_4 = p$ in Corollary 3, and raising the inequality to power $p + 1$, we have

$$T(L_{m,p+2}; x, y)^{p+1} \cdot T(L_{m,p}; x, y)^{p+1} \geqslant T(L_{m,p+1}; x, y)^{2(p+1)}, \tag{9}$$

while the inductive hypothesis gives

$$T(L_{m,p+1}; x, y)^{p} \geqslant T(L_{m,p}; x, y)^{p+1}. \tag{10}$$

The result follows by multiplying (9) and (10). □

Note that because of symmetry in the definition of $a_{m,n}(x, y)$ (see (3a)), Proposition 15 also implies the sequence $(a_{m,n}(x, y)\colon\ m \in \mathbb{Z}_{>0})$ is monotonically increasing for a fixed $n \in \mathbb{Z}_{>0}$.

Proposition 16. *The sequence $(b_n(x, y)\colon\ n \in \mathbb{Z}_{>0})$ is monotonically increasing.*

Proof. From Proposition 15, for all $n \in \mathbb{Z}_{>0}$,

$$b_{n+1}(x, y) = a_{n+1,n+1}(x, y) \geqslant a_{n+1,n}(x, y) \geqslant a_{n,n}(x, y) = b_n(x, y). \qquad \square$$

Proposition 17. *The sequence $(\alpha_m(x, y)\colon\ m \in \mathbb{Z}_{>0})$ is monotonically increasing and converges to the limit $\beta(x, y)$.*

Proof. From Proposition 15, for all $m, n \in \mathbb{Z}_{>0}$, $a_{m+1,n}(x, y) \geqslant a_{m,n}(x, y)$. Taking the $n \to \infty$ limit on both sides, we get for all $m \in \mathbb{Z}_{>0}$, $\alpha_{m+1}(x, y) \geqslant \alpha_m(x, y)$. That is, the sequence is monotonically increasing.

Now, if $m \in \mathbb{Z}_{>0}$ then from Proposition 15 we know that for all integers $n \geqslant m$, $a_{m,n}(x, y) \leqslant a_{n,n}(x, y) = b_n(x, y)$. Thus, taking the $n \to \infty$ limit on both sides, for all $m \in \mathbb{Z}_{>0}$, $\alpha_m(x, y) \leqslant \beta(x, y)$, which leads to

$$\lim_{m\to\infty} \alpha_m(x, y) \leqslant \beta(x, y). \tag{11}$$

Also for all integers $n \geqslant m$, Proposition 15 implies $a_{m,n}(x, y) \geqslant a_{m,m}(x, y) = b_m(x, y)$. Taking the $n \to \infty$ limit on both sides, we get for all $m \in \mathbb{Z}_{>0}$, $\alpha_m(x, y) \geqslant b_m(x, y)$, which in turn implies

$$\lim_{m\to\infty} \alpha_m(x, y) \geqslant \lim_{m\to\infty} b_m(x, y) = \beta(x, y). \tag{12}$$

The result follows from (11) and (12). □

The following restatement of the convergence in the previous result is worth noting.

Corollary 18.

$$\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n}(x, y) = \lim_{n\to\infty}\lim_{m\to\infty} a_{m,n}(x, y) = \lim_{n\to\infty} b_n(x, y).$$

Since the terms of the double sequence $(a_{m,n}(x, y)\colon\ m, n \in \mathbb{Z}_{>0})$ are monotonically increasing, both for a fixed $m \in \mathbb{Z}_{>0}$ and a fixed $n \in \mathbb{Z}_{>0}$, it follows that its iterated limits shown above are also equal to its double limit [8, Theorem I].

Corollary 19.

$$\lim_{m,n\to\infty} a_{m,n}(x, y) = \lim_{n\to\infty} b_n(x, y).$$

Lemma 20. *For fixed $k, m \in \mathbb{Z}_{>0}$, the sequence $(c^k_{m,n}(x, y)\colon n \in \mathbb{Z}_{>k})$ is monotonically increasing.*

Proof. From (5a) it is enough to show, for all integers $n > 1$,

$$T(L_{m,n+1}; x, y)^{n-k} \cdot T(L_{m,k}; x, y) \geqslant T(L_{m,n}; x, y)^{n-k+1}. \tag{13}$$

Our proof is by induction on n. When $n = k + 1$, we get (13) by using $n_1 = k + 2$, $n_2 = n_3 = k + 1$ and $n_4 = k$ in Corollary 3. Let $p \in \mathbb{Z}_{>k}$, and suppose (13) is true for all $n \leqslant p$.

Now let $n = p + 1$. Then from the inductive hypothesis we have,

$$T(L_{m,p+1}; x, y)^{p-k} \cdot T(L_{m,k}; x, y) \geqslant T(L_{m,p}; x, y)^{p-k+1}, \tag{14}$$

while using $n_1 = p + 2$, $n_2 = n_3 = p + 1$ and $n_4 = p$ in Corollary 3, we get

$$T(L_{m,p+2}; x, y) \cdot T(L_{m,p}; x, y) \geqslant T(L_{m,p+1}; x, y)^2.$$

Raising the last inequality to the power $p - k + 1$, we see

$$T(L_{m,p+2}; x, y)^{p-k+1} \cdot T(L_{m,p}; x, y)^{p-k+1} \geqslant T(L_{m,p+1}; x, y)^{2(p-k+1)}, \tag{15}$$

and the result follows by multiplying (14) and (15). □

Lemma 21. *For fixed $m, n \in \mathbb{Z}_{>0}$ such that $n > 2$, the finite sequence $(c^k_{m,n}(x, y)\colon 0 < k < n - 1)$ is monotonically increasing.*

Proof. It is enough to prove for integers k such that $0 < k < n - 1$,

$$T(L_{m,n}; x, y) \cdot T(L_{m,k}; x, y)^{n-k-1} \geqslant T(L_{m,k+1}; x, y)^{n-k}. \tag{16}$$

We use induction on k. When $k = n - 2$, (16) follows from Corollary 3 using $n_1 = n$, $n_2 = n_3 = n - 1$ and $n_4 = n - 2$. Now, let p be an integer in the interval $1 < p < n - 1$, and assume (16) is true for all k in the interval $p \leqslant k < n - 1$.

Suppose $k = p - 1$. Then, from the inductive hypothesis, we have

$$T(L_{m,n}; x, y) \cdot T(L_{m,p}; x, y)^{n-p-1} \geqslant T(L_{m,p+1}; x, y)^{n-p}, \tag{17}$$

Also, using $n_1 = p + 1$, $n_2 = n_3 = p$ and $n_4 = p - 1$ in Corollary 3, we have

$$T(L_{m,p+1}; x, y) \cdot T(L_{m,p-1}; x, y) \geqslant T(L_{m,p}; x, y)^2.$$

Raising the last inequality to the power $n - p$, we obtain

$$T(L_{m,p+1}; x, y)^{n-p} \cdot T(L_{m,p-1}; x, y)^{n-p} \geqslant T(L_{m,p}; x, y)^{2(n-p)}. \tag{18}$$

The result follows by multiplying (17) and (18). □

We are unable to verify if, for a fixed $k, n \in \mathbb{Z}_{>0}$ such that $n > k$, the sequence $(c^k_{m,n}(x, y)\colon m \in \mathbb{Z}_{>0})$ is monotonically increasing.

Lemma 22. *For a fixed $k \in \mathbb{Z}_{>0}$, the sequence $(\gamma^k_n(x, y)\colon n \in \mathbb{Z}_{>k})$ is monotonically increasing and converges to the limit $\beta(x, y)$.*

Proof. From Lemma 20, for any $m, n \in \mathbb{Z}_{>0}$ such that $n > k$, we know $c^k_{m,n+1}(x, y) \geqslant c^k_{m,n}(x, y)$. Taking the $m \to \infty$ limit on both sides and applying Lemma 13, we get $\gamma^k_{n+1}(x, y) \geqslant \gamma^k_n(x, y)$.

That the sequence converges to $\beta(x, y)$ is straightforward from (5b) and Proposition 17. □

Lemma 23. *For a fixed $n \in \mathbb{Z}_{>2}$, the finite sequence $(\gamma^k_n(x, y)\colon 0 < k < n - 1)$ is monotonically increasing.*

Proof. From Lemma 21, for any $m \in \mathbb{Z}_{>0}$ and k in the interval $0 < k < n-1$, $c_{m,n}^{k+1}(x, y) \geqslant c_{m,n}^{k}(x, y)$. The result follows by taking the $m \to \infty$ limits on both sides of the inequality and applying Lemma 13. □

4.2. Computing lower bounds for $\alpha_m(x, y)$ and $\beta(x, y)$

One consequence of our results is that for a fixed $m \in \mathbb{Z}_{>0}$, each term in the sequence $(a_{m,n}(x, y)\colon\ n \in \mathbb{Z}_{>0})$ is a lower bound for $\alpha_m(x, y)$. Similarly, for fixed $k, m \in \mathbb{Z}_{>0}$, every term of the sequence $(c_{m,n}^k(x, y)\colon\ n \in \mathbb{Z}_{>0})$ is also a lower bound for $\alpha_m(x, y)$. Among these two sequences, we now show that for fixed $k, m \in \mathbb{Z}_{>0}$ the terms of $(c_{m,n}^k(x, y)\colon\ n \in \mathbb{Z}_{>0})$ give better lower bounds for $\alpha_m(x, y)$ than the corresponding terms in $(a_{m,n}(x, y)\colon\ n \in \mathbb{Z}_{>0})$.

Lemma 24. *For fixed $k, m, n \in \mathbb{Z}_{>0}$ such that $n > k$, $c_{m,n}^k(x, y) \geqslant a_{m,n}(x, y)$.*

Proof. A routine manipulation shows this is equivalent to $a_{m,n}(x, y) \geqslant a_{m,k}(x, y)$, which is true from Proposition 15. □

Of course, unlike $a_{m,n}(x, y)$, to compute $c_{m,n}^k(x, y)$, we also need $T(L_{m,k}; x, y)$ for some integer k in the interval $0 < k < n$, in addition to $T(L_{m,n}; x, y)$. Also, from Lemma 21, it is better to choose k as large as possible in the interval $0 < k < n$ when using $c_{m,n}^k(x, y)$ as a lower bound for $\alpha_m(x, y)$. However, even if no such $T(L_{m,k}; x, y)$ is available, since $T(L_{m,1}; x, y) = x^{m-1}$, we can always compute $c_{m,n}^1(x, y)$ as a better lower bound for $\alpha_m(x, y)$ than $a_{m,n}(x, y)$. For a fixed $m \in \mathbb{Z}_{>0}$, these observations help us to compute a lower bound for $\alpha_m(x, y)$ if $T(L_{m,n}; x, y)$ is known for some fixed integer $n > 1$.

Theorem 25. *For fixed $k, m, n \in \mathbb{Z}_{>0}$ such that $n > k$,*

$$\alpha_m(x, y) \geqslant \left(\frac{T(L_{m,n}; x, y)}{T(L_{m,k}; x, y)}\right)^{1/m(n-k)}.$$

Proof. Follows from Lemmas 12 and 20, and (5a). □

From Proposition 17 and Lemma 22 we know that $\alpha_n(x, y)$ and $\gamma_n^k(x, y)$, for fixed $k, n \in \mathbb{Z}_{>0}$ such $k < n$, are lower bounds for $\beta(x, y)$. Of these, we now show that $\gamma_n^k(x, y)$ is a better lower bound for $\beta(x, y)$ than $\alpha_n(x, y)$.

Lemma 26. *For a fixed $k, n \in \mathbb{Z}_{>0}$ such that $n > k$, $\gamma_n^k(x, y) \geqslant \alpha_n(x, y)$.*

Proof. Take the $m \to \infty$ limits on both sides of the inequality in Lemma 24. □

It also follows from Lemma 23 that it is better to choose k as large as possible in $\gamma_n^k(x, y)$ when using it as a lower bound for $\beta(x, y)$. In summary, we get the following lower bound for $\beta(x, y)$ in $H_{>1}$.

Theorem 27. *For fixed $k, n, p, q \in \mathbb{Z}_{>0}$ such that $n > k$ and $p > q$,*

$$\beta(x, y) \geqslant \alpha_k(x, y)^{-k/(n-k)} \cdot \left(\frac{T(L_{n,p}; x, y)}{T(L_{n,q}; x, y)}\right)^{1/(n-k)(p-q)}.$$

Proof. From Proposition 17, we know

$$\beta(x, y) \geqslant \gamma_n^k(x, y) = \alpha_k(x, y)^{-k/(n-k)} \cdot \alpha_n(x, y)^{n/(n-k)}.$$

The result now can be deduced from Theorem 25. □

Observe that to apply Theorem 27 to compute a lower bound for $\beta(x,y)$, we need the exact value (or an upper bound) for $\alpha_k(x,y)$, for some k in the interval $0<k<n$. It is easy to verify from (4a) and (3a) that $\alpha_1(x,y)=x$ for all $x,y\geqslant 1$. Additionally, from Shrock [18, p. 439], we get for all $x,y\geqslant 1$,

$$\alpha_2(x,y)=\sqrt{\frac{1+y+x+x^2+\sqrt{y^2+2y(1+x-x^2)+(x^2+x+1)^2}}{2}}, \tag{19}$$

while Chang and Shrock [5] also give a method to compute $\alpha_3(x,y)$ exactly for all $x,y\geqslant 1$. Hence, in practice, we can use $k\in\{1,2,3\}$ in Theorem 27 to compute a lower bound for $\beta(x,y)$ in $H_{>1}$.

Finally, it is worth remembering from Lemma 11 that we can further improve our computation of a lower bound for $\beta(x,y)$ by choosing the maximum of computed bounds for both $\beta(x,y)$ and $\beta(y,x)$.

We illustrate our results in this region using the following known value $\beta(3,3)=(\Gamma(1/4)/2\Gamma(3/4))^4=4.78926\ldots$ (see Baxter [1, Table 1]), where $\Gamma(t)$, $t\in\mathbb{R}$, is the Gamma function. Using the publicly available software to compute Tutte polynomials by Haggard, Pearce and Royle [11], we obtain $T(L_{6,6};3,3)=4.420\times 10^{21}$ and $T(L_{5,6};3,3)=6.101\times 10^{17}$, while (19) gives $\alpha_2(3,3)=\sqrt{8+\sqrt{37}}$. Now, substituting these values and $k=2$, $n=p=6$, $q=5$ in Theorem 27, we find $\beta(3,3)\geqslant 4.76242$.

4.3. Further remarks on computing $\alpha_m(x,y)$

For a fixed $m\in\mathbb{Z}_{>0}$, we can use Theorem 25 to compute a lower bound for $\alpha_m(x,y)$, if $T(L_{m,n};x,y)$ is known for some integer $n>1$. However, when m is large (usually $m>10$), computing $T(L_{m,n};x,y)$ is often unwieldy even for small integers $n>2$. In this section, we show that it is also possible to compute a lower bound for $\alpha_m(x,y)$ from $T(L_{p,n};x,y)$ for some $p,n\in\mathbb{Z}_{>1}$ such that $p\leqslant m$.

First, it is instructive to note a somewhat straightforward solution to this problem. From $T(L_{p,n};x,y)$, we can compute a lower bound for $\alpha_p(x,y)$ using Theorem 25. As $p\leqslant m$, from Proposition 17, clearly such a lower bound for $\alpha_p(x,y)$ is also a lower bound for $\alpha_m(x,y)$. However, we now show that $\xi_m^{p,k}(x,y)$ (defined in (7)) can often improve upon this straightforward lower bound.

Lemma 28. *For fixed $k,m,p\in\mathbb{Z}_{>0}$ such that $k<p\leqslant m$, $\alpha_m(x,y)\geqslant\xi_m^{p,k}(x,y)$. Furthermore, if $mk\geqslant pq$, where $q\equiv m \pmod{p-k}$ and $k\leqslant q\leqslant p-1$, then $\xi_m^{p,k}(x,y)\geqslant\alpha_p(x,y)$.*

Proof. A routine manipulation using (7) and (5b) shows that the first inequality is equivalent to $\gamma_m^q(x,y)\geqslant\gamma_p^k(x,y)$, which is clearly true from Lemmas 22 and 23.

To prove the second inequality, note that it is enough to show

$$\alpha_p(x,y)^{mk-pq}\cdot\alpha_q(x,y)^{q(p-k)}\geqslant\alpha_k(x,y)^{k(m-q)}.$$

When $mk\geqslant pq$, from Proposition 17, we have

$$\alpha_p(x,y)^{mk-pq}\cdot\alpha_q(x,y)^{q(p-k)}\geqslant\alpha_q(x,y)^{mk-pq+q(p-k)}\geqslant\alpha_k(x,y)^{k(m-q)}. \qquad \square$$

We do not know if $\xi_m^{p,k}(x,y)\geqslant\alpha_p(x,y)$ when $mk<pq$. In practice, if $mk<pq$, we may simply use the maximum value of $\xi_m^{p,k}(x,y)$ and $\alpha_p(x,y)$ (or, of their lower bounds) as a lower bound for $\alpha_m(x,y)$.

If we are able to choose the integers k,m,p in the above lemma such that $p-k\mid m-k$, then we have the following corollary, which can be easily deduced from Lemma 28 by setting $q=k$.

Corollary 29. *For fixed $k,m,p\in\mathbb{Z}_{>0}$ such that $k<p\leqslant m$ and $p-k\mid m-k$,*

$$\alpha_m(x,y)\geqslant\left(\frac{\alpha_p(x,y)^{p(m-k)}}{\alpha_k(x,y)^{k(m-p)}}\right)^{1/m(p-k)}\geqslant\alpha_p(x,y).$$

For a fixed $k, m, p \in \mathbb{Z}_{>0}$ such that $k < p < m$, it also appears difficult to determine if $\xi_m^{p+1,k}(x, y) \geqslant \xi_m^{p,k}(x, y)$, or if $\xi_m^{p,k}(x, y) \geqslant \xi_m^{p,k-1}(x, y)$ when $k > 1$. If true, these inequalities would validate the intuition that it is better to choose p and k as large as possible in Lemma 28 to obtain a lower bound for $\alpha_m(x, y)$. Another open question is if $\xi_m^{p,k}(x, y) \geqslant \xi_{m-1}^{p,k}(x, y)$ for fixed $k, m, p \in \mathbb{Z}_{>0}$ such that $k < p < m$, which would justify computing $\xi_m^{p,k}(x, y)$ as a lower bound for $\alpha_m(x, y)$ instead of $\xi_{m'}^{p,k}(x, y)$ for some m' in the interval $p < m' < m$. Nevertheless, we next show that for a fixed $k, p \in \mathbb{Z}_{>0}$ such that $p > k$, the sequence $(\xi_m^{p,k}(x, y)\colon m \in \mathbb{Z}_{\geqslant p})$ has a monotonically increasing infinite subsequence.

Lemma 30. *For fixed $k, p \in \mathbb{Z}_{>0}$ such that $p > k$, the sequence $(\xi_{n(p-k)+k}^{p,k}(x, y)\colon n \in \mathbb{Z}_{>0})$ is monotonically increasing.*

Proof. From (7), for fixed $n \in \mathbb{Z}_{>0}$,

$$\xi_{n(p-k)+k}^{p}(x, y) = \left(\frac{\alpha_p(x, y)^{pn}}{\alpha_k(x, y)^{k(n-1)}}\right)^{1/(n(p-k)+k)}.$$

Hence, a routine check shows that

$$\xi_{(n+1)(p-k)+k}^{p,k}(x, y) \geqslant \xi_{n(p-k)+k}^{p,k}(x, y),$$

is equivalent to $\alpha_p(x, y) \geqslant \alpha_k(x, y)$, which is true from Proposition 17. □

In summary, Lemma 28 and Theorem 25 give us the following lower bound for $\alpha_m(x, y)$ for a fixed $m \in \mathbb{Z}_{>0}$, which is particularly useful when computing $T(L_{m,n}; x, y)$ is difficult for integers $n > 1$.

Theorem 31. *For fixed $k, m, p \in \mathbb{Z}_{>0}$ such that $k < p < m$, if $q \equiv m \pmod{p-k}$ with $k \leqslant q \leqslant p - 1$, then for fixed $r, s, t, u \in \mathbb{Z}_{>0}$ such that $r > s$ and $t > u$,*

$$\alpha_m(x, y) \geqslant \left(\alpha_k(x, y)^{-k} \cdot \left(\frac{T(L_{p,r}; x, y)}{T(L_{p,s}; x, y)}\right)^{1/(r-s)}\right)^{(m-q)/m(p-k)} \left(\frac{T(L_{q,t}; x, y)}{T(L_{q,u}; x, y)}\right)^{1/m(t-u)}.$$

Proof. From Lemma 28 and (7), we know

$$\alpha_m(x, y) \geqslant \alpha_k(x, y)^{-k(m-q)/m(p-k)} \cdot \alpha_p(x, y)^{p(m-q)/m(p-k)} \cdot \alpha_q(x, y)^{q/m}.$$

The result now follows by applying Theorem 25 to obtain lower bounds for both $\alpha_p(x, y)$ and $\alpha_q(x, y)$ on the right hand side of this inequality. □

For $k, m, p \in \mathbb{Z}_{>0}$ such that $k < p < m$ and $(p - k) \mid (m - k)$, we can use Corollary 29 in the place of Lemma 28, along with Theorem 25, to obtain the following.

Corollary 32. *Let $k, m, p \in \mathbb{Z}_{>0}$ such that $k < p < m$ and $p - k \mid m - k$. Then for fixed $r, s \in \mathbb{Z}_{>0}$ such that $r > s$,*

$$\alpha_m(x, y) \geqslant \alpha_k(x, y)^{-k(m-p)/m(p-k)} \cdot \left(\frac{T(L_{p,r}; x, y)}{T(L_{p,s}; x, y)}\right)^{(m-k)/m(p-k)(r-s)}.$$

5. Results in $H_{<1}$

Throughout this section, we assume $(x, y) \in H_{<1}$.

In contrast to our results in the previous section, it appears to be difficult to show that the sequences $(a_{m,n}(x, y)\colon n \in \mathbb{Z}_{>0})$ for a fixed $m \in \mathbb{Z}_{>0}$, and $(b_n(x, y)\colon n \in \mathbb{Z}_{>0})$, are monotonically increasing when $(x, y) \in H_{<1}$.

5.1. Monotonicity and convergence results

It is easy to see that Theorem 8 is equivalent to Corollary 3, when $n_4 = 1$ and the direction of inequality is reversed. Hence, not surprisingly, many of our results on the sequences $(c^1_{m,n}(x, y)\colon m, n \in \mathbb{Z}_{>0},\ n > 1)$ and $(\gamma^1_n(x, y)\colon n \in \mathbb{Z}_{>1})$ in this section are analogous to the results obtained in Section 4.1 specialized to $k = 1$, with the direction of inequality reversed.

Lemma 33. *For a fixed $m \in \mathbb{Z}_{>0}$, the sequence $(c^1_{m,n}(x, y)\colon n \in \mathbb{Z}_{>1})$ is monotonically decreasing.*

We skip the proof of Lemma 33, which is identical to the proof of Lemma 20 specialized for $k = 1$, using Theorem 8 instead of Corollary 3, and the direction of inequality reversed in each step.

We are unable to verify if, for a fixed $n \in \mathbb{Z}_{>1}$, the sequence $(c^1_{m,n}(x, y)\colon m \in \mathbb{Z}_{>0})$ is monotonically increasing.

Lemma 34. *The sequence $(\gamma^1_n(x, y)\colon n \in \mathbb{Z}_{>1})$ is monotonically decreasing, and*

$$\lim_{n\to\infty} \gamma^1_n(x, y) \geqslant \beta(x, y).$$

Proof. The proof that the sequence is monotonically decreasing is identical to that of Lemma 22 specialized to $k = 1$, and the direction of inequality reversed in each step.

We next show that each term of the sequence $(\gamma^1_n(x, y)\colon n \in \mathbb{Z}_{>1})$ is bounded below by $\beta(x, y)$.

Let $n \in \mathbb{Z}_{>1}$. Choose an integer m such that $n - 1 \mid m - 1$, and consider the lattice $L_{m,m}$. Then there exists a non-negative integer r such that $m = (n - 1)r + 1$. Let i be an integer in the interval $0 \leqslant i < r$. Applying Theorem 8 with $n_1 = m - i(n - 1)$, $n_2 = n$, $n_3 = m - (i + 1)(n - 1)$ and $n_4 = 1$, we get

$$T(L_{m,m-i(n-1)}; x, y) \cdot T(L_{m,1}; x, y) \leqslant T(L_{m,n}; x, y) \cdot T(L_{m,m-(i+1)(n-1)}; x, y). \tag{20}$$

Multiplying together (20) for all integers i in the interval $0 \leqslant i < r$, we obtain

$$T(L_{m,m}; x, y) \cdot T(L_{m,1}; x, y)^{r-1} \leqslant T(L_{m,n}; x, y)^r.$$

Substituting $r = (m - 1)/(n - 1)$ in the above inequality, we find

$$T(L_{m,m}; x, y) \cdot T(L_{m,1}; x, y)^{(m-n)/(n-1)} \leqslant T(L_{m,n}; x, y)^{(m-1)/(n-1)}.$$

Raising both sides of the inequality to the power $1/m^2$ and taking the $m \to \infty$ limits, we get

$$\beta(x, y) \cdot \alpha_1(x, y)^{1/(n-1)} \leqslant \alpha_n(x, y)^{n/n-1}.$$

Thus, from (5b), $\beta(x, y) \leqslant \gamma^1_n(x, y)$.

In other words, every element of the monotonic decreasing sequence $(\gamma^1_n(x, y)\colon n \in \mathbb{Z}_{>1})$ is bounded below by $\beta(x, y)$. Hence the sequence is convergent [13, p. 36], and

$$\lim_{n\to\infty} \gamma^1_n(x, y) \geqslant \beta(x, y). \qquad \square$$

Observe that, unlike Lemma 22, we are only able to show the limit of the sequence $(\gamma^1_n(x, y)\colon n \in \mathbb{Z}_{>1})$ is bounded below by $\beta(x, y)$ in the region $H_{<1}$. This is because we do not have a result analogous to Proposition 17 in this region.

Lastly, note that at the point $(1, 1)$, using Corollary 10 instead of Theorem 8, both the results in this section can be strengthened to use an arbitrary $k \in \mathbb{Z}_{>0}$ instead of $k = 1$.

5.2. Computing upper bounds for $\alpha_m(x,y)$ and $\beta(x,y)$

Lemmas 12 and 33 imply that for a fixed $m \in \mathbb{Z}_{>0}$, each term of the sequence $(c^1_{m,n}(x,y)\colon n \in \mathbb{Z}_{>1})$ is an upper bound for $\alpha_m(x,y)$. Similarly, from Lemma 34, every term of the sequence $(\gamma^1_n(x,y)\colon n \in \mathbb{Z}_{>1})$ is an upper bound for $\beta(x,y)$.

We note that at the point $(1,1)$, where we can select any integer $k \in \mathbb{Z}_{>0}$ to compute upper bounds for $\alpha_m(1,1)$ and $\beta(1,1)$, it is indeed better to choose k as large as possible. That is for fixed $k,m,n \in \mathbb{Z}_{>0}$ such that $n > k$, we have both $c^{k+1}_{m,n}(1,1) \leqslant c^k_{m,n}(1,1)$ and $\gamma^{k+1}_n(1,1) \leqslant \gamma^k_n(1,1)$. We omit the proof of these observations, which are identical to Lemmas 21 and 23 specialized to the point $(x=1, y=1)$, and the direction of inequality reversed in each step.

Summarizing our results on computing upper bounds for $\alpha_m(x,y)$ and $\beta(x,y)$ in $H_{<1}$, we get the following results from Lemmas 33 and 34. We again skip the proofs that are identical to Theorems 25 and 27, respectively, with the direction of inequality reversed.

Theorem 35.

1. *For fixed $m,n \in \mathbb{Z}_{>0}$ such that $n > 1$,*

$$\alpha_m(x,y) \leqslant \left(\frac{T(L_{m,n};x,y)}{x^{m-1}}\right)^{1/m(n-1)}.$$

2. *For fixed $k,m,n \in \mathbb{Z}_{>0}$ such that $n > k$,*

$$\alpha_m(1,1) \leqslant \left(\frac{T(L_{m,n};1,1)}{T(L_{m,k};1,1)}\right)^{1/m(n-k)}.$$

Theorem 36.

1. *For fixed $n,p \in \mathbb{Z}_{>1}$,*

$$\beta(x,y) \leqslant x^{-1/(n-1)} \cdot \left(\frac{T(L_{n,p};x,y)}{x^{n-1}}\right)^{1/(n-1)(p-1)}.$$

2. *For fixed k,n,p,q such that $n > k$ and $p > q$,*

$$\beta(1,1) \leqslant \alpha_k(1,1)^{-k/(n-k)} \cdot \left(\frac{T(L_{n,p};1,1)}{T(L_{n,q};1,1)}\right)^{1/(n-k)(p-q)}.$$

It is worth remembering from Lemma 11 that we can further improve our computation of an upper bound for $\beta(x,y)$ by choosing the minimum of computed bounds for both $\beta(x,y)$ and $\beta(y,x)$.

In [4] it was shown that $3.64497 \leqslant \beta(2,1) \leqslant 3.74101$. We can now improve upon their upper bound using the value $T(L_{7,7};2,1) = 1.316 \times 10^{24}$ from [4, Table 1], along with $n = p = 7$ in Theorem 36-1 to get $\beta(2,1) \leqslant 3.71224$. Further, using the value $T(L_{14,14};2,1) = 9.203 \times 10^{103}$ from [20], and $n = p = 14$ in Theorem 36-1 gives us the following (apparently) current best upper bound for this limit.

Corollary 37.

$$\beta(2,1) \leqslant 3.705603.$$

As some evidence for the closeness of the bounds obtainable from the above results to the exact values of the corresponding limits, we mention that at the point $(1,1)$, substituting the values $T(L_{25,25};1,1) = 9.599 \times 10^{296}$, $T(L_{25,24};1,1) = 5.093 \times 10^{284}$, $\alpha_2(1,1) = \sqrt{2+\sqrt{3}}$, $k = 2$, $n = p = 25$ and $q = 24$ in Theorem 36-2 gives $\beta(1,1) \leqslant 3.227297$, while the exact value computed for this limit in [22] is 3.209912.... (Note that $T(L_{25,25}; 1, 1)$ and $T(L_{25,24}; 1, 1)$ can be easily computed using the matrix-tree theorem [12, Chapter 1], while we get $\alpha_2(1, 1)$ from (19).)

5.3. *Further remarks on computing $\alpha_m(x, y)$*

Our observations in this section mirror those in Section 4.3.

For our next result, we use (7) to define $\xi_m^{p,k}(x, y)$ for $(x, y) \in H_{<1}$ and $k, m, p \in \mathbb{Z}_{>0}$ such that $k < p \leqslant m$.

Lemma 38. *For fixed $m, p \in \mathbb{Z}_{>1}$ such that $p \leqslant m$, $\alpha_m(x, y) \leqslant \xi_m^{p,1}(x, y)$.*

Our proof of Lemma 38 is identical to the proof of the first inequality in Lemma 28 using Theorem 8 in the place of Corollary 3, and the direction of the inequality reversed in each step.

As in the region $H_{>1}$, we are unable to verify if $\xi_m^{p+1,1}(x, y) \leqslant \xi_m^{p,1}(x, y)$ and $\xi_m^{p,1}(x, y) \leqslant \xi_{m-1}^{p,1}(x, y)$, for a fixed $p, m \in \mathbb{Z}_{>0}$ such that $1 < p < m$. We can, however, show the following, analogous to Lemma 30, and we skip the identical proof specialized to $k = 1$ with the direction of inequality reversed.

Lemma 39. *For a fixed $p \in \mathbb{Z}_{>1}$, the sequence $(\xi_{n(p-1)+1}^{p,1}(x, y)\colon n \in \mathbb{Z}_{>0})$ is monotonically decreasing.*

As before, at the point $(1, 1)$, the above statements can be generalized to any integer $k \in \mathbb{Z}_{>0}$ using Corollary 10.

We summarize these observations as follows. Their proofs are identical to Theorem 31 and Corollary 32 with the direction of the inequality reversed in each step, and are omitted.

Theorem 40.

1. *For fixed $m, p \in \mathbb{Z}_{>1}$ such that $p < m$, if $q \equiv m \pmod{p-1}$ with $1 \leqslant q \leqslant p-1$, then for fixed $r, t \in \mathbb{Z}_{>1}$,*

$$\alpha_m(x, y) \leqslant \left(x^{-1}\left(\frac{T(L_{p,r}; x, y)}{x^{p-1}}\right)^{1/(r-1)}\right)^{(m-q)/m(p-1)}\left(\frac{T(L_{q,t}; x, y)}{x^{q-1}}\right)^{1/m(t-1)}.$$

2. *For fixed $k, m, p \in \mathbb{Z}_{>0}$ such that $k < p < m$, if $q \equiv m \pmod{p-k}$ with $k \leqslant q \leqslant p-1$, then for all fixed $r, s, t, u \in \mathbb{Z}_{>0}$ such that $r > s$ and $t > u$,*

$$\alpha_m(1, 1) \leqslant \left(\alpha_k(1, 1)^{-k} \cdot \left(\frac{T(L_{p,r}; 1, 1)}{T(L_{p,s}; 1, 1)}\right)^{1/(r-s)}\right)^{(m-q)/m(p-k)}\left(\frac{T(L_{q,t}; 1, 1)}{T(L_{q,u}; 1, 1)}\right)^{1/m(t-u)}.$$

Corollary 41.

1. *Let $m, p \in \mathbb{Z}_{>1}$ such that $p < m$ and $p - 1 \mid m - 1$. Then for a fixed $r \in \mathbb{Z}_{>1}$,*

$$\alpha_m(x, y) \leqslant x^{-(m-p)/m(p-1)} \cdot \left(\frac{T(L_{p,r}; x, y)}{x^{p-1}}\right)^{(m-1)/m(p-1)(r-1)}.$$

2. *Let $k, m, p \in \mathbb{Z}_{>0}$ such that $k < p < m$ and $p - k \mid m - k$. Then for fixed $r, s \in \mathbb{Z}_{>0}$ such that $r > s$,*

$$\alpha_m(1, 1) \leqslant \alpha_k(1, 1)^{-k(m-p)/m(p-k)} \cdot \left(\frac{T(L_{p,r}; 1, 1)}{T(L_{p,s}; 1, 1)}\right)^{(m-k)/m(p-k)(r-s)}.$$

6. Cylindrical square lattices

In this section, we briefly show that some of our results for the lattice graph $L_{m,n}$ in the region $H_{>1}$ can also be extended to square lattices embedded on a cylindrical surface. For $m, n \in \mathbb{Z}_{>0}$, a *cylindrical square lattice* is the graph $C_{m,n}$ with vertex set $V = V(L_{m,n})$ and edge set $E = E(L_{m,n}) \cup \{\{(1, j), (m, j)\}\colon 1 \leqslant j \leqslant n\}$, where $V(L_{m,n})$ and $E(L_{m,n})$ are the vertex and edge sets of $L_{m,n}$, respectively.

We omit the proof of the next result which is similar to the proof of Lemma 1.

Lemma 42. *Let $m, n_1, n_2, n_3 \in \mathbb{Z}_{>0}$ such that $n_1 = n_2 + n_3$. Then for all $x, y \geqslant 1$,*

$$T(C_{m,n_1}; x, y) \geqslant T(C_{m,n_2}; x, y) \cdot T(C_{m,n_3}; x, y).$$

We skip the proof of the next result which is similar to Corollary 3 and is also a consequence of Theorem 2.

Lemma 43. *Let m, n_1, n_2, n_3, n_4 be positive integers such that $n_1 \geqslant \max\{n_2, n_3\}$ and $n_1 + n_4 = n_2 + n_3$. Then for all $x, y \geqslant 1$ such that $(x - 1)(y - 1) > 1$,*

$$T(C_{m,n_1}; x, y) \cdot T(C_{m,n_4}; x, y) \geqslant T(C_{m,n_2}; x, y) \cdot T(C_{m,n_3}; x, y).$$

For $m, n \in \mathbb{Z}_{>0}$ and $(x, y) \in \mathbb{R}^2$, let

$$d_{m,n}(x, y) = T(C_{m,n}; x, y)^{1/mn} \quad \text{and} \tag{21}$$

$$\eta_m(x, y) = \lim_{n\to\infty} d_{m,n}(x, y). \tag{22}$$

Additionally, for $k, m, n \in \mathbb{Z}_{>0}$ such that $k < n$ and $(x, y) \in \mathbb{R}^2$, let

$$f^k_{m,n}(x, y) = \left(\frac{T(C_{m,n}; x, y)}{T(C_{m,k}; x, y)}\right)^{1/m(n-k)}. \tag{23}$$

Note that it can be easily shown that for all $x, y \geqslant 1$,

$$\lim_{m\to\infty} d_{m,n}(x, y) = \lim_{m\to\infty} a_{m,n}(x, y) = \alpha_n(x, y) \quad \text{and}$$
$$\lim_{n\to\infty} d_{n,n}(x, y) = \lim_{n\to\infty} b_n(x, y) = \beta(x, y).$$

Our next result can be readily verified from Lemmas 42 and 43, using arguments similar to those in Section 4.

Theorem 44. *Let $(x, y) \in H_{>1}$.*

1. *For a fixed $m \in \mathbb{Z}_{>0}$, the sequence $(d_{m,n}(x, y)\colon n \in \mathbb{Z}_{>0})$ is monotonically increasing.*
2. *The sequence $(\eta_m(x, y)\colon m \in \mathbb{Z}_{>0})$ is monotonically increasing and converges to the limit $\beta(x, y)$.*
3. *For fixed $k, m \in \mathbb{Z}_{>0}$, the sequence $(f^k_{m,n}(x, y)\colon n \in \mathbb{Z}_{>k})$ is monotonically increasing and converges to the limit $\eta_m(x, y)$.*
4. *For fixed $m, n \in \mathbb{Z}_{>0}$ such that $n > 2$, the finite sequence $(f^k_{m,n}(x, y)\colon 0 < k < n - 1)$ is monotonically increasing.*
5. *For fixed $k, m, n \in \mathbb{Z}_{>0}$ such that $n > k$,*

$$\eta_m(x, y) \geqslant \left(\frac{T(C_{m,n}; x, y)}{T(C_{m,k}; x, y)}\right)^{1/m(n-k)}.$$

We do not yet know of an inequality similar to Theorem 8 for the graph $C_{m,n}$ in the region $(x, y) \in H_{<1}$, and the behavior of the sequence $(f^k_{m,n}(x, y)\colon n \in \mathbb{Z}_{>k})$ in this region, for fixed $k, m \in \mathbb{Z}_{>0}$, is an open problem.

7. Open questions

We used a family of Tutte polynomial inequalities to study the analytical properties of the sequences $(a_{m,n}(x, y)\colon m, n \in \mathbb{Z}_{>0})$ and $(b_n(x, y)\colon n \in \mathbb{Z}_{>0})$, and compute one-sided bounds for their limits in the region $x, y \geqslant 1$. However, several theoretical and computational questions remain open.

Theoretically, in the region $H_{<1}$ it is unknown if the sequences $(a_{m,n}(x, y)\colon n \in \mathbb{Z}_{>0})$ for a fixed $m \in \mathbb{Z}_{>0}$, $(b_n(x, y)\colon n \in \mathbb{Z}_{>0})$ and $(\alpha_m(x, y)\colon m \in \mathbb{Z}_{>0})$, are monotonically increasing. Also, except for the point $(1, 1)$, it is also open if the sequence $(c^k_{m,n}(x, y)\colon n \in \mathbb{Z}_{>k})$ for fixed $k \in \mathbb{Z}_{>1}$, $m \in \mathbb{Z}_{>0}$, is monotonically decreasing in $H_{<1}$. We can solve this problem if we obtain a result analogous to Corollary 3, with the direction of inequality reversed, for the region $H_{<1}$. Another significant unanswered question in this region is if the sequences $(\alpha_m(x, y)\colon m \in \mathbb{Z}_{>0})$ and $(\gamma^1_n(x, y)\colon n \in \mathbb{Z}_{>0})$ converge to the limit $\beta(x, y)$ (see Lemma 34).

Computationally, in both regions $H_{>1}$ and $H_{<1}$, we do not have a rigorous analysis of the closeness of our bounds to the corresponding limits $\alpha_m(x, y)$ and $\beta(x, y)$. For fixed $k, m, n \in \mathbb{Z}_{>0}$ such that $n > k$ and $x, y \geqslant 1$, let $\Delta^{n,k}_m(x, y) = |\alpha_m(x, y) - c^k_{m,n}(x, y)|$ and $\Delta^{n,k}(x, y) = |\beta(x, y) - \gamma^k_n(x, y)|$. An asymptotic analysis of $\Delta^{n,k}_m(x, y)$ and $\Delta^{n,k}(x, y)$ as a function of n and k could provide a basis for an efficient approximation algorithm for computing the limits $\alpha_m(x, y)$ and $\beta(x, y)$. A related unanswered question is the rate of convergence of the various sequences to their respective limits as a function of $(x - 1)(y - 1)$. It would be of interest to analyze the behavior of both $\Delta^{n,k}_m(x, y)$ and $\Delta^{n,k}(x, y)$ as a function of x and y, for fixed $k, m, n \in \mathbb{Z}$ with $n > k$.

Note that, unlike the transfer matrix methods used in [3] and [4], our approach only provides one-sided bounds to the limits $\alpha_m(x, y)$ and $\beta(x, y)$. We do not know if it is possible to obtain similar inequalities and non-trivial upper bounds for these limits in the region $H_{>1}$, and lower bounds in $H_{<1}$.

Finally, it would also be interesting to know if these results can be extended to points in other regions of the two-dimensional real plane. In particular, for the region $0 \leqslant x < 1$, $y \geqslant 0$, or $x \geqslant 0$, $0 \leqslant y < 1$ where the Tutte polynomial is known to be positive, it is still open if there are inequalities similar to those in Section 2.

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Appendix A. Proof of Lemma 7

We begin with the following planar embedding of $L_{m,n}$ on the two-dimensional real plane. A vertex (i, j) of $L_{m,n}$, where $1 \leqslant i \leqslant m$ and $1 \leqslant j \leqslant n$, corresponds to the point in $\mathbb{R}^2$ with integer coordinates (i, j). An edge between vertices (i, j) and $(i + 1, j)$ is a straight line connecting the corresponding points whenever $i \leqslant i \leqslant m - 1$. Similarly, an edge between (i, j) and $(i, j + 1)$ is a straight line connecting the corresponding points whenever $i \leqslant j \leqslant n - 1$. Henceforth, in this proof, we consider $L_{m,n}$ to be a plane graph with such a planar embedding.

It is straightforward to check that the lemma is true if any of the sets P_1, P_2 or R is empty. Assume for a contradiction that the lemma is false. Then there exists minimal sets $P_1 \subseteq E_1 \setminus E_2$,

$P_2 \subseteq E_2 \setminus E_1$, $R \subseteq E_k$, and a minor $J \in \mathcal{MF}(L_{m,n}, E_k \setminus R)$, such that P_1 and P_2 are not R-submodular in J.

Then, from (6), $J = G/C \setminus D$ for some disjoint sets $C, D \subseteq E_k \setminus R$ such that $C \cup D = E_k \setminus R$. Consider the following planar embedding of J obtained from $L_{m,n}$. Beginning with $L_{m,n}$, for every $e \in E_k \setminus R$, we do the following.

- If $e \in D$, we simply delete the line corresponding to the edge e.
- If $e \in C$ and connects the vertices (i,k) and $(i+1,k)$, where $1 \leqslant i \leqslant m-1$, we delete the line corresponding to e and identify its two end points to form a single vertex at (i,k).

It is easy to see that the resulting graph J is also a plane graph. Let ρ be the rank function of the graph J.

We now make the following claim for P_1, P_2, R and J.

Claim 45.

1. *The subgraph $\langle P_1 \cup P_2 \rangle$ of J is acyclic.*
2. *For $i \in \{1,2\}$, every $e \in R$ is in a cycle in the subgraph $\langle P_i \cup R \rangle$ of J.*

Proof. (1) Assume for a contradiction that subgraph $\langle P_1 \cup P_2 \rangle$ contains a cycle. Without loss of generality, then there is an edge $p \in P_1$ which is in a cycle. By the minimality condition, we know the sets $P_1' = P_1 \setminus \{p\}$ and P_2 are R-submodular in J. That is, there exists a bijection $\pi : 2^R \to 2^R$ such that for all $C \subseteq R$,

$$\rho\big(P_1' \cup P_2 \cup C\big) + \rho(R \setminus C) \leqslant \rho\big(P_1' \cup \pi C\big) + \rho(P_2 \cup R \setminus \pi C).$$

Since p is in a cycle in $P_1 \cup P_2$, we know $\rho(P_1 \cup P_2 \cup C) = \rho(P_1' \cup P_2 \cup C)$, and hence π is also an R-submodular bijection of (P_1, P_2) in J, which is a contradiction.

(2) We prove the case $i = 1$ below. The case $i = 2$ is analogous.

Assume for a contradiction that there exists an $e \in R$ such that e is a bridge in $\langle P_1 \cup R \rangle$, and let $R' = R \setminus \{e\}$. Clearly, by our choice of P_1, P_2 and R, P_1 and P_2 are R'-submodular in $J \setminus e$. That is, there exists a $\pi' : 2^{R'} \to 2^{R'}$ such that for all $C \subseteq R'$,

$$\rho(P_1 \cup P_2 \cup C) + \rho\big(R' \setminus C\big) \leqslant \rho\big(P_1 \cup \pi' C\big) + \rho\big(P_2 \cup R' \setminus \pi' C\big).$$

Since e is a bridge in $P_1 \cup R$, this implies that for all $C \subseteq R'$,

$$\rho(P_1 \cup P_2 \cup C) + \rho\big(R' \cup \{e\} \setminus C\big) \leqslant \rho\big(P_1 \cup \pi' C \cup \{e\}\big) + \rho\big(P_2 \cup R' \setminus \pi' C\big). \tag{24}$$

Similarly, by our choice, the sets P_1 and P_2 are also R'-submodular in J/e. Hence, there exists a bijection $\pi'' : 2^{R'} \to 2^{R'}$ such that for all $C \subseteq R'$,

$$\rho\big(P_1 \cup P_2 \cup C \cup \{e\}\big) + \rho\big(R' \cup \{e\} \setminus C\big) \leqslant \rho\big(P_1 \cup \pi'' C \cup \{e\}\big) + \rho\big(P_2 \cup R' \cup \{e\} \setminus \pi'' C\big).$$

Once again, as e is a bridge in $P_1 \cup R$, we have for all $C \subseteq R'$,

$$\rho\big(P_1 \cup P_2 \cup C \cup \{e\}\big) + \rho\big(R' \setminus C\big) \leqslant \rho\big(P_1 \cup \pi'' C\big) + \rho\big(P_2 \cup R' \cup \{e\} \setminus \pi'' C\big). \tag{25}$$

It follows from (24) and (25) that the bijection $\pi : 2^R \to 2^R$ such that for all $C \subseteq R$,

$$\pi C = \begin{cases} \pi' C \cup \{e\}, & \text{if } e \notin C; \\ \pi''(C \setminus \{e\}), & \text{if } e \in C, \end{cases}$$

is an R-submodular bijection between P_1 and P_2 in J, which is a contradiction. $\square$

Now, let Q be a maximal path in J with edges from R. Clearly, Q is a straight line connecting points (i,k) and (j,k) for some i, j such that $1 \leqslant i \leqslant j \leqslant m$ in the plane graph J. Also, all vertices in subgraph $\langle P_1 \cup Q \rangle$ of J are points (a, b) where $1 \leqslant a \leqslant m$ and $1 \leqslant b \leqslant k$. From Claim 45-2, we also know that every edge $q \in Q$ is in a cycle in the plane subgraph $\langle P_1 \cup Q \rangle$ of J. It follows that there is a path that is edge-disjoint with Q connecting the end points of Q in the subgraph $\langle P_1 \cup Q \rangle$ of J. Analogously, there is a similar path edge-disjoint with Q in the subgraph $\langle P_2 \cup Q \rangle$ of J. Together, this means that $P_1 \cup P_2$ contains a cycle in J, which contradicts Claim 45-1. $\square$

References

[1] R.J. Baxter, Critical antiferromagnetic square-lattice Potts model, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 383 (1784) (1982) 43–54.
[2] N.L. Biggs, Chromatic and thermodynamic limits, J. Phys. A 8 (10) (1975) L110–L112.
[3] N.L. Biggs, Colouring square lattice graphs, Bull. Lond. Math. Soc. 9 (1) (1977) 54–56.
[4] N. Calkin, C. Merino, S. Noble, M. Noy, Improved bounds for the number of forests and acyclic orientations in the square lattice, Electron. J. Combin. 10 (2003) #R4.
[5] S.-C. Chang, R. Shrock, Exact Potts model partition functions on wider arbitrary-length strips of the square lattice, Phys. A 296 (2001) 234–288.
[6] S.-C. Chang, R. Shrock, Tutte polynomials and related asymptotic limiting functions for recursive families of graphs, Adv. in Appl. Math. 32 (2004) 44–87.
[7] S.-C. Chang, R. Shrock, Some exact results for spanning trees on lattices, J. Phys. A 39 (2006) 5653–5658.
[8] E.L. Dodd, On iterated limits of multiple sequences, Math. Ann. 61 (1) (1905) 95–108.
[9] T. Feder, M. Mihail, Balanced matroids, in: Proceedings of the 24th Annual ACM Symposium on Theory of Computing (STOC), Victoria, British Columbia, Canada, ACM Press, New York, 1992, pp. 26–38.
[10] G.R. Grimmett, The rank polynomials of large random lattices, J. Lond. Math. Soc. 18 (2) (1978) 567–575.
[11] G. Haggard, D.J. Pearce, G. Royle, Computing Tutte polynomials, ACM Trans. Math. Software 37 (3) (2010) 24:1–24:17.
[12] M. Jerrum, Counting, Sampling and Integrating: Algorithms and Complexity, Lectures Math. ETH Zurich, Birkhäuser, 2003.
[13] S. Lang, Undergraduate Analysis, second edition, Undergrad. Texts Math., Springer, 1997.
[14] A.P. Mani, Some inequalities for Whitney–Tutte polynomials, Combin. Probab. Comput. 19 (3) (2010) 425–439.
[15] C. Merino, D.J.A. Welsh, Forests, colorings and acyclic orientations of the square lattice, Ann. Comb. 3 (1999) 417–429.
[16] J.G. Oxley, Matroid Theory, Oxford University Press, 1992.
[17] P. Seymour, D.J.A. Welsh, Combinatorial applications of an inequality from statistical mechanics, Math. Proc. Cambridge Philos. Soc. 77 (1975) 485–495.
[18] R. Shrock, Exact Potts model partition functions on ladder graphs, Phys. A 283 (2000) 388–446.
[19] R. Shrock, F.Y. Wu, Spanning trees on graphs and lattices in d dimensions, J. Phys. A 33 (2000) 3881–3902.
[20] P. Tittmann, Enumeration in graphs, Webpage. http://www.htwm.de/peter/research/enumeration.html (Accessed November 2009).
[21] D.J.A. Welsh, Complexity: Knots, Colourings and Counting, London Math. Soc. Lecture Note Ser., vol. 186, Cambridge University Press, 1993.
[22] F.Y. Wu, Number of spanning trees on a lattice, J. Phys. A 10 (6) (1977) L113–L115.